COMPLEX LAGRANGIAN EMBEDDINGS OF MODULI SPACES OF VECTOR BUNDLES

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Abstract. By means of a Fourier-Mukai transform we embed moduli spaces $\mathcal{M}_C(r,d)$ of stable bundles on an algebraic curve C of genus $g(C) \geq 2$ as isotropic subvarieties of moduli spaces of μ-stable bundles on the Jacobian variety $J(C)$. When $g(C) = 2$ this provides new examples of special Lagrangian submanifolds.

1. Introduction

Throughout this paper we shall fix $\mathbb{C}$ as the ground field. Let C be a smooth algebraic curve of genus $g > 1$, and denote by $J(C)$ its Jacobian variety and by $\Theta \in H^2(J(C), \mathbb{Z})$ the cohomology class corresponding to the theta divisor. Fix coprime positive integers r, d such that $d > 2rg$, and let $\mathcal{M}_C(r,d)$ be the moduli space of stable vector bundles on C of Chern character (r,d). We show that $\mathcal{M}_C(r,d)$ can be embedded as an isotropic holomorphic submanifold of the complex symplectic variety $\mathcal{M}^{\mu}_{J(C)}(r,d) = \mathcal{M}^{\mu}_{J(C)}(d + r(1-g), -r\Theta, 0, \ldots, 0)$ — the moduli space of μ-stable vector bundles on $J(C)$ with Chern character $(d + r(1-g), -r\Theta, 0, \ldots, 0)$ (cf. Theorem 2.1 for a precise statement). When $g(C) = 2$ one has $\dim \mathcal{M}^{\mu}_{J(C)}(r,d) = 2 \dim \mathcal{M}_C(r,d)$, and by using the hyper-Kähler structure of $\mathcal{M}^{\mu}_{J(C)}(r,d)$, one can choose on this space a complex structure such that $\mathcal{M}_C(r,d)$ embeds as a special Lagrangian submanifold, thus providing new examples of such objects.

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We recall a few facts about the Fourier-Mukai transform in the context of Abelian varieties [8]. Let X be an Abelian variety and $\widehat{X} = \mathrm{Pic}^0(X)$ its dual variety. Let $\mathcal{P}$ be the normalized Poincaré bundle on $X \times \widehat{X}$. The Mukai functor is defined as

$$\mathbf{R}\mathcal{S}\colon D(X) \to D(\widehat{X})$$
$$\mathbf{R}\mathcal{S}(-) = \mathbf{R}\pi_{\widehat{X},*}(\pi_X^*(-) \otimes \mathcal{P})$$

where $D(X)$ and $D(\widehat{X})$ are the bounded derived categories of coherent sheaves on X and $\widehat{X}$, respectively. Mukai has shown that the functor $\mathbf{R}\mathcal{S}$ is invertibile and preserves families of sheaves (cf. [8, 10]). If E is a WIT_i sheaf on X, that is, a sheaf whose transform is concentrated in degree i, then the functor $\mathbf{R}\mathcal{S}$ preserves the Ext groups:

$$\mathrm{Ext}^j_X(E, E) \cong \mathrm{Ext}^j_{\widehat{X}}(\hat{E}, \hat{E}) \quad \text{for every } j,$$

where $\hat{E}$ indicates the transform of E.

Let C be a smooth projective curve of genus $g > 1$ and $J(C)$ the Jacobian of C. If we fix a base point x_0 on C, and let $\alpha_{x_0}\colon C \to J(C)$ be the Abel-Jacobi embedding given by $\alpha_{x_0}(x) = \mathcal{O}_C(x - x_0)$, the normalized Poincaré bundle $\mathcal{P}_C$ on $C \times J(C)$ is the pullback of the Poincaré bundle on $J(C) \times J(C)$, where we identify $J(C)$ with $\widehat{J(C)}$ via the isomorphism $-\phi_\Theta\colon J(C) \to \widehat{J(C)}$. The Poincaré bundle on $C \times J(C)$ gives rise to a derived functor (which is not invertible):

$$\mathbf{R}\Phi_C\colon D(C) \to D(J(C))$$
$$\mathbf{R}\Phi_C(-) = \mathbf{R}\pi_{J(C),*}(\pi_C^*(-) \otimes \mathcal{P}_C)\,.$$

Since α_{x_0} is a closed immersion we have a natural isomorphism of functors

$$\mathbf{R}\Phi_C \cong \mathbf{R}\mathcal{S} \circ \alpha_{x_0,*}. \tag{1}$$

Thus the study of the transforms of bundles F on C with respect to $\mathbf{R}\Phi_C$ is equivalent to studying the transforms of sheaves of pure dimension 1 of the form $\alpha_{x_0,*}(F)$ with respect to $\mathbf{R}\mathcal{S}$. We recall the following fact which is proven in [6].

Proposition 1.1. *If E is a stable bundle on C of rank r and degree d such that $d > 2rg$, then E is* WIT_0*, and the transformed sheaf $\hat{E} =$*

$\mathbf{R}^0\Phi_C(E)$ *is locally free and* μ*-stable with respect to the theta divisor on* $J(C)$*.*

2. Complex Lagrangian embeddings

If we consider the moduli space $\mathcal{M}_C(r,d)$ of stable bundles of rank r and degree d on a projective smooth curve of genus $g > 1$ such that $d > 2rg$ and r, d are coprime, the functor $\mathbf{R}\Phi_C$ gives rise to an injective morphism

$$\tilde{\jmath}\colon \mathcal{M}_C(r,d) \to \mathcal{M}^{\mu}_{J(C)}(r,d) = \mathcal{M}^{\mu}_{J(C)}(d + r(1-g), -r\Theta, 0, \dots, 0)$$

where the sheaves in $\mathcal{M}^{\mu}_{J(C)}(r,d)$ are stable with respect to the polarization Θ.

Before studying the morphism $\tilde{\jmath}$ we need to recall some elementary facts about the Yoneda product of Ext groups. Let $\mathcal{A}$ be an abelian category with enough injectives. The elements of $\operatorname{Ext}^1_{\mathcal{A}}(E,E)$ are identified with equivalence classes of exact sequences $0 \to E \to F \to E \to 0$ with respect to the usual relation. This can be generalized to the groups $\operatorname{Ext}^2_{\mathcal{A}}(E,E)$ as follows. We refer to [2] for proofs and details.

Consider the following commutative diagram with exact rows:

(2)
$$\begin{array}{ccccccccccc}
E: & 0 & \longrightarrow & B & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \mathrm{Id}_B} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}_A} & & \\
E': & 0 & \longrightarrow & B & \longrightarrow & G'_1 & \longrightarrow & G'_2 & \longrightarrow & A & \longrightarrow & 0.
\end{array}$$

We write $E \twoheadrightarrow E'$ when such a diagram holds. The relation $\twoheadrightarrow$ is not symmetric, but it generates the following equivalence relation: $E \sim E'$ if and only if there exists a chain of sequences $E_0, E_1, \dots, E_k$ such that

$$E = E_0 \twoheadrightarrow E_1 \twoheadleftarrow E_2 \twoheadrightarrow \cdots \twoheadleftarrow E_k = E'.$$

Let $\operatorname{Yext}^2_{\mathcal{A}}(-,-)$ the set of such equivalence classes.

There is an isomorphism

$$\operatorname{Yext}^2_{\mathcal{A}}(-,-) \cong \operatorname{Ext}^2_{\mathcal{A}}(-,-).$$

From now on we shall identify the above groups. Observe that the identity of $\mathrm{Ext}^2_{\mathcal{A}}(A,B)$ is given by the class of the sequence

$$0 \longrightarrow B \xrightarrow{\mathrm{Id}_B} B \xrightarrow{0} A \xrightarrow{\mathrm{Id}_A} A \longrightarrow 0.$$

Moreover the Yoneda product

$$\mathrm{Ext}^1_{\mathcal{A}}(B,A) \times \mathrm{Ext}^1_{\mathcal{A}}(A,C) \to \mathrm{Ext}^2_{\mathcal{A}}(B,C)$$

is obtained in the following way: let E and E' be two elements of $\mathrm{Ext}^1_{\mathcal{A}}(B,A)$ and $\mathrm{Ext}^1_{\mathcal{A}}(A,C)$ represented respectively by the sequences

$$E: \quad 0 \longrightarrow A \xrightarrow{\nu} F \xrightarrow{p} B \longrightarrow 0$$

$$E': \quad 0 \longrightarrow C \xrightarrow{i} G \xrightarrow{\lambda} A \longrightarrow 0.$$

Then the class of the exact sequence

$$0 \longrightarrow C \xrightarrow{i} G \xrightarrow{\nu\circ\lambda} F \xrightarrow{p} B \longrightarrow 0$$

in $\mathrm{Ext}^2_{\mathcal{A}}(B,C)$ is the image of E, E' with respect to the Yoneda product.

We shall also need to introduce a moduli space of stable sheaves in Simpson's sense [13]. For simplicity we denote the Abel-Jacobi map as $j\colon C \to J(C)$. Observe that if E is a stable bundle on C then $j_*(E)$ is a stable sheaf of pure dimension 1 on $J(C)$ with respect to the polarization Θ. Let $\mathcal{M}^{\mathrm{pure}}_{J(C)}(r,d)$ be the moduli space of all stable pure sheaves on $J(C)$ with Chern character $(0,\dots,0,r\Theta,d+r(1-g))$. If $\mathcal{E}$ is a flat family of vector bundles on C parametrized by a Noetherian scheme S, then $j_{S,*}(\mathcal{E})$ is a flat family of sheaves on $J(C)\times S$ over S, where $j_S\colon C\times S \to J(C)\times S$ is the embedding $j\times \mathrm{Id}_S$. Therefore one has a morphism of moduli spaces

$$j_* : \mathcal{M}(r,d) \to \mathcal{M}^{\mathrm{pure}}_{J(C)}(r,d)\,. \tag{3}$$

Lemma 2.1. *The morphism $\tilde{\jmath}\colon \mathcal{M}_C(r,d) \to M^{\mu}_{J(C)}(r,d)$ is an immersion (i.e., its tangent map is injective).*

Proof. From the isomorphism given by Eq. (1) and recalling that the transform $\mathbf{R}\mathcal{S}$ preserves the Ext groups of WIT sheaves, it is enough to show that the same claim holds for the morphism (3). By the very

definition of the Kodaira-Spencer map, the tangent map to j_* may be identified with the map

$$\mathrm{Ext}^1_C(E,E) \overset{\phi}{\hookrightarrow} \mathrm{Ext}^1_{J(C)}(j_*(E), j_*(E))$$

obtained in the following way. Let

$$A: \quad 0 \longrightarrow E \longrightarrow F \longrightarrow E \longrightarrow 0 \tag{4}$$

be a sequence representing an element of $\mathrm{Ext}^1_C(E,E)$. If we apply the functor j_* to the above sequence we obtain the exact sequence

$$B: \quad 0 \longrightarrow j_*(E) \longrightarrow j_*(F) \longrightarrow j_*(E) \longrightarrow 0. \tag{5}$$

One checks immediately that the map $\phi([A]) = [B]$ is well defined. If $\phi([A]) = 0$ then $\phi([A])$ is represented by the extension

$$0 \longrightarrow j_*(E) \longrightarrow j_*(E) \oplus j_*(E) \longrightarrow j_*(E) \longrightarrow 0. \tag{6}$$

Now applying the functor j^* to the above sequence and noting that $j^*(j_*(H)) \cong H$ for every vector bundle H on C we obtain the split exact sequence

$$0 \longrightarrow E \longrightarrow E \oplus E \longrightarrow E \longrightarrow 0. \tag{7}$$

Therefore $\phi([A]) = 0$ implies $[A] = 0$ and ϕ is injective. $\square$

Mukai proved that the moduli space of simple sheaves on an abelian surface X is symplectic; more precisely, the Yoneda pairing

$$\upsilon \colon \mathrm{Ext}^1_X(E,E) \times \mathrm{Ext}^1_X(E,E) \to \mathrm{Ext}^2_X(E,E) \cong \mathbb{C}$$

defines a holomorphic symplectic form on the moduli of simple sheaves on X (cf. [9, 11]). When $\dim X = 2n > 2$ to define a symplectic form on the smooth locus of the moduli space one needs to choose a symplectic form ω on X. The symplectic form is then defined by the compositions (cf. [5])

$$\begin{aligned} \mathrm{Ext}^1_X(E,E) \times \mathrm{Ext}^1_X(E,E) &\to \mathrm{Ext}^2_X(E,E) \xrightarrow{\mathrm{tr}} H^2(X, \mathcal{O}_X) \\ &\overset{\sim}{\to} H^{0,2}(X,\mathbb{C}) \xrightarrow{\lambda} H^{n,n}(X,\mathbb{C}) \cong \mathbb{C} \end{aligned} \tag{8}$$

where tr is the trace morphisms and the map λ is obtained by wedging by $\omega^n \wedge \bar{\omega}^{n-1}$.

Theorem 2.1. *If $g(C)$ is even, and the map $\tilde{\jmath}$ embeds $\mathcal{M}(r,d)$ into the smooth locus $\mathcal{M}^0_{J(C)}(r,d)$ of $\mathcal{M}^\mu_{J(C)}(r,d)$, then the subvarieties $\mathcal{M}_C(r,d)$ are isotropic with respect to any of the symplectic forms defined by equation (8). In particular, when $g(C)=2$ the subvarieties $\mathcal{M}_C(r,d)$ are Lagrangian with respect to the Mukai form of $\mathcal{M}^\mu_{J(C)}(r,d)$.*

Proof. Since $\mathcal{M}^0_{J(C)}(r,d)$ is smooth, and $\tilde{\jmath}\colon \mathcal{M}(r,d) \to \mathcal{M}^0_{J(C)}(r,d)$ is injective and is an immersion, it is also an embedding. Now, let $E \in \mathcal{M}_C(r,d)$. It is enough to show that the Yoneda product

$$\begin{aligned}\operatorname{Ext}^1_{J(C)}(j_*(E), j_*(E)) \quad \times \quad & \operatorname{Ext}^1_{J(C)}(j_*(E), j_*(E)) \\ & \longrightarrow \operatorname{Ext}^2_{J(C)}(j_*(E), j_*(E))\end{aligned}$$

vanishes when applied to pairs $([A],[B])$ of elements in $\operatorname{Ext}^1_{J(C)}(j_*(E), j_*(E))$ where $[A]$ and $[B]$ are represented, respectively, by the sequences

$$A: \qquad 0 \longrightarrow j_*(E) \xrightarrow{\nu} j_*(F) \xrightarrow{p} j_*(E) \longrightarrow 0$$

$$B: \qquad 0 \longrightarrow j_*(E) \xrightarrow{i} j_*(G) \xrightarrow{\lambda} j_*(E) \longrightarrow 0$$

with $F, G \in \mathcal{M}_C(r,d)$. It is enough to remark that the product of the classes of the sequences of sheaves on C

$$0 \to E \to F \to E \to 0\,, \qquad 0 \to E \to G \to E \to 0$$

is zero for dimensional reasons, and apply the functor j_*.

In the case $g(C)=2$ the moduli space is smooth by the results in [9]; moreover,

$$\dim \mathcal{M}^\mu_{J(C)}(r,d) = 2(r^2+1) = 2\dim \mathcal{M}_C(r,d)\,.$$

□

Remark 3. If we consider the moduli space $\mathcal{M}_C(r,\xi)$ of stable bundles on C of rank r and fixed determinant isomorphic to ξ, then the result is trivial: the variety $\mathcal{M}_C(r,\xi)$ is Fano, so that it carries no holomorphic forms. ▲

4. The case $g(C)=2$

In this section we elaborate on the case $g(C)=2$. One can characterize situations where the moduli space $\mathcal{M}^{\mu}_{J(C)}(r,d)$ is compact. This happens for instance in the following case.

Proposition 4.1. *Assume $g(C)=2$, $d>4r$ and that $\rho=d-r$ is a prime number. Then every Gieseker-semistable sheaf on $J(C)$ with Chern character $(d-r,-r\Theta,0)$ is μ-stable. Moreover, if $d>r^2+r$, every such sheaf is locally free (this always happens when $r=1,2,3$).*

Proof. Since $d-r$ is prime, every sheaf in $\mathcal{M}_{J(C)}(r,d)$ is properly stable. Let $[F]\in\mathcal{M}_{J(C)}(r,d)$ and assume that the subsheaf G destabilizes F. Let $\mathrm{ch}(G)=(\sigma,\xi,s)$. Standard computations show that if F is not μ-stable then

$$\frac{\xi\cdot\Theta}{\sigma}=-\frac{2r}{\rho}\qquad\text{and}\qquad s<0\,.$$

Setting $n=\xi\cdot\Theta$ we have $|n|=2r\sigma/\rho$, with $\sigma<\rho$ and $\rho>3r$. This is impossible whenever ρ is prime.

The statement about local freeness follows from the Bogomolov inequality. □

In the case $g(C)=2$ the complex Lagrangian embedding $\tilde{\jmath}\colon\mathcal{M}_C(r,d)\to\mathcal{M}^{\mu}_{J(C)}(r,d)$ provides new examples of *special Lagrangian submanifolds.* We refer to [1, 7] for the definition and the main properties of these objects. Now, if X is a hyper-Kähler manifold of complex dimension $2n$, let I, J, K be three complex structures compatible with the hyper-Kähler metric, and such that $IJ=K$. Let ω_I, ω_J, ω_K be the corresponding Kähler forms. Then the 2-form $\Omega=\omega_I+i\omega_J$ is a holomorphic symplectic form in the complex structure K. It is easy to check that a K-complex n-dimensional submanifold which is Lagrangian with respect to Ω is special Lagrangian in the structure J [3].

One should notice that via the Hitchin-Kobayashi correspondence (which identifies μ-stable bundles on a Kähler manifold with irreducible Einstein-Hermite bundles, cf. [5]), the space $\mathcal{M}^{\mu}_{J(C)}(r,d)$ acquires a hyper-Kähler structure, compatible with a Kähler form provided by

the Weil-Petersson metric, and with a holomorphic symplectic form which may be identified with the Mukai form [4].

Therefore we obtain the following result.

Proposition 4.2. *The space $\mathcal{M}^{\mu}_{J(C)}(r,d)$ has a complex structure such that $\tilde{\jmath}\colon \mathcal{M}_C(r,d) \to \mathcal{M}^{\mu}_{J(C)}(r,d)$ is a special Lagrangian submanifold.*

The elements of the Jacobian variety $J(C)$ act on the embedding $j\colon C \to J(C)$ by translation, so that for every $x \in J(C)$ we have a special Lagrangian submanifold $\tilde{\jmath}_x\colon \mathcal{M}_C(r,d) \to \mathcal{M}^{\mu}_{J(C)}(r,d)$. This provides a family of deformations of $\tilde{\jmath}(\mathcal{M}_C(r,d))$ through special Lagrangian submanifolds. As one easily shows, this embeds $J(C)$ into the moduli space $\mathcal{M}_{SL}$ of special Lagrangian deformations of $\tilde{\jmath}(\mathcal{M}_C(r,d))$ (notice that $\dim_{\mathbb{R}} \mathcal{M}_{SL} = b_1(\mathcal{M}_C(r,d)) = 4 = \dim_{\mathbb{R}}(J(C)))$ [12]. The case $r=1$ is somehow trivial because $\mathcal{M}^{\mu}_{J(C)}(1,d) \simeq J(C) \times J(C)$ by a result of Mukai [8].

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